

# SOME RULES FOR POLYDIMENSIONAL SQUEEZING

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## Abstract

The review of the following results of the Refs. [1] - [5] is presented: For mixed state light of  $N$ -mode electromagnetic field described by Wigner function which has generic Gaussian form the photon distribution function is obtained and expressed explicitly in terms of Hermite polynomials of  $2N$ -variables. The momenta of this distribution are calculated and expressed as functions of matrix invariants of the dispersion matrix. The role of new uncertainty relation depending on photon state mixing parameter is elucidated. New sum rules for Hermite polynomials of several variables are found. The photon statistics of polymode even and odd coherent light and squeezed polymode Schrödinger cat light is given explicitly. Photon distribution for polymode squeezed number states expressed in terms of multivariable Hermite polynomials is discussed.

## 1 Introduction

In the Ref.[1] it was shown that the matrix elements of density matrix in number state basis for polymode oscillator are expressed in terms of Hermite polynomials of several variables for the density operator in the canonically transformed thermal state of the oscillator. In the recent works [2], [3] the photon distribution function for the generic Gaussian light described by the Wigner function which is the most generic Gaussian in quadrature phase space was found and expressed in terms of Hermite polynomials of 2 variables for one-mode case [2] and  $2N$  variables for polymode case [3]. The physical meaning of mixed Gaussian state of the light may be understood if one takes into account that the pure multimode Gaussian state corresponds to the generalised correlated state introduced by Sudarshan [6] who related those states to the symplectic dynamical group. The mixed Gaussian states studied above may be considered as the mixture of generalised correlated states plus thermal noise acting on each mode which has its own temperature. The photon distribution function for even and odd coherent states [7] or Schrödinger cat states [8] subject to squeezing both in one-mode and polymode cases has been found in Ref.[4]. The polymode Schrödinger cat states and photon distributions for light in these states were introduced in [5]. The aim of this work is to give a review of the photon distribution functions and related sum rules for one-mode and polymode Gaussian light and for the even and odd coherent states light using the results of [2] - [5]. It should be emphasized that the squeezed light is worth to be used in interferometric gravitational antennas [9] and the even and odd coherent light may play alternative role in gravitational wave experiment [10],[11].

## 2 Photon Distribution for Polymode Gaussian Light

The mixed squeezed state of the  $N$ -mode light with a *Gaussian* density operator  $\hat{\rho}$  is described by the Wigner function (see, for example, [12])

$$W(\mathbf{p}, \mathbf{q}) = (\det \mathbf{M})^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} (\mathbf{Q} - \langle \mathbf{Q} \rangle) \mathbf{M}^{-1} (\mathbf{Q} - \langle \mathbf{Q} \rangle) \right], \quad (1)$$

where  $2N$ -dimensional vector  $\mathbf{Q} = (\mathbf{p}, \mathbf{q})$  consists of  $N$  components  $p_1, \dots, p_N$  and  $N$  components  $q_1, \dots, q_N$ .  $2N$  parameters  $\langle p_i \rangle$  and  $\langle q_i \rangle$ ,  $i = 1, 2, \dots, N$ , combined into vector  $\langle \mathbf{Q} \rangle$ , are the average values of the quadratures. A real symmetric quadrature dispersion matrix  $\mathbf{M}$  has  $2N^2 + N$  parameters

$$\mathcal{M}_{\alpha\beta} = \frac{1}{2} \langle \hat{Q}_\alpha \hat{Q}_\beta + \hat{Q}_\beta \hat{Q}_\alpha \rangle - \langle \hat{Q}_\alpha \rangle \langle \hat{Q}_\beta \rangle, \quad \alpha, \beta = 1, 2, \dots, 2N. \quad (2)$$

They obey certain constraints, which are nothing but the generalized uncertainty relations [12]. The photon distribution function in this state has the form [3], [1]

$$\mathcal{P}_{\mathbf{n}} = \mathcal{P}_0 \frac{H_{\mathbf{n}\mathbf{n}}^{(\mathbf{R})}(\mathbf{y})}{\mathbf{n}!}, \quad (3)$$

where vector  $\mathbf{n}$  consists of  $N$  nonnegative integers:  $\mathbf{n} = (n_1, n_2, \dots, n_N)$ . The function  $H_{\mathbf{n}\mathbf{n}}^{(\mathbf{R})}(\mathbf{y})$  is the Hermite polynomial of  $2N$  variables. We introduced also notations

$$\mathbf{n}! = n_1! n_2! \dots n_N!. \quad (4)$$

The symmetric  $2N$ -dimensional matrix  $\mathbf{R}$  and the  $2N$ -dimensional vector  $\mathbf{y}$  are given by the relations

$$\mathbf{R} = \mathbf{U}^\dagger (\mathbf{I}_{2N} - 2\mathbf{M}) (\mathbf{I}_{2N} + 2\mathbf{M})^{-1} \mathbf{U}^*, \quad (5)$$

$$\mathbf{y} = 2\mathbf{U}^\dagger (\mathbf{I}_{2N} - 2\mathbf{M})^{-1} \langle \mathbf{Q} \rangle, \quad (6)$$

where the  $2N$ -dimensional unitary matrix

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\mathbf{I}_N & i\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{I}_N \end{pmatrix}$$

is introduced. The matrices  $\mathbf{I}_N$  and  $\mathbf{I}_{2N}$  are identity matrices of corresponding dimensions.

The probability to have no photons has the form

$$\mathcal{P}_0 = \left[ \det \left( \mathbf{M} + \frac{1}{2} \mathbf{I}_{2N} \right) \right]^{-\frac{1}{2}} \exp \left[ -\langle \mathbf{Q} \rangle (\mathbf{M} + \frac{1}{2} \mathbf{I}_{2N})^{-1} \langle \mathbf{Q} \rangle \right]. \quad (7)$$

It may be shown [3], [4] that the multivariable Hermite polynomial is the even function if the sum of its indices is the even number and the polynomial with the odd sum of indices is the odd function. Due to this parity property of the polydimensional Hermite polynomials the "diagonal" multivariable Hermite polynomial is the even function since the sum of its indices is always even

number. Consequently the above photon distribution function is the even function. Thus the photon distribution function for generic mixed Gaussian light found in [3] is expressed in terms of multivariable Hermite polynomials and it depends on the quadrature means and dispersions. The photon number means and dispersion matrix corresponding to the found distribution (3) are of the form

$$\begin{aligned} \langle n_j \rangle &= \frac{1}{2}(\sigma_{p_j p_j} + \sigma_{q_j q_j} - 1) + \frac{1}{2}(\langle p_j \rangle^2 + \langle q_j^2 \rangle), \\ \sigma_{n_j n_j} &= \frac{1}{2}(T_j^2 - 2d_j - \frac{1}{2}) + \langle \mathbf{Q}_j \rangle \mathcal{M}_j \langle \mathbf{Q}_j \rangle, \end{aligned}$$

where  $T_j$  and  $d_j$  are the trace and the determinant of the  $2 \times 2$ -matrix  $\mathcal{M}_j$ , describing only  $j$ -th mode, and the 2-vector  $\mathbf{Q}_j$  has the components  $(p_j, q_j)$ .

### 3 Pure Polymode States

The photon distribution for polymode squeezed correlated state may be expressed in terms of symplectic transform parameters relating boson operators as follows

$$\begin{pmatrix} \hat{\mathbf{b}} \\ \hat{\mathbf{b}}^\dagger \end{pmatrix} = \Omega \begin{pmatrix} \hat{\mathbf{a}} \\ \hat{\mathbf{a}}^\dagger \end{pmatrix} + \begin{pmatrix} \mathbf{d} \\ \mathbf{d}^* \end{pmatrix}, \quad \Omega = \begin{pmatrix} \zeta & \eta \\ \eta^* & \zeta^* \end{pmatrix}, \quad (8)$$

where  $\Omega$  is a symplectic  $2N \times 2N$ -matrix consisting of four  $N$ -dimensional complex square blocks, and  $\mathbf{d}$  is a complex  $N$ -vector. Then we have for photon distribution in squeezed correlated polymode state  $|\beta\rangle$  labeled by the complex number vector with  $N$ -components

$$\mathcal{P}_{\mathbf{n}} = \frac{\mathcal{P}_0(\beta)}{\mathbf{n}!} \left| H_{\mathbf{n}}^{\{\zeta^{-1}\eta\}} \left( \eta^{-1}[\beta - \mathbf{d}] \right) \right|^2, \quad \mathcal{P}_0(\beta) = |\mathcal{F}_0(\beta)|^2 \exp -|\beta|^2, \quad (9)$$

where

$$\mathcal{F}_0(\beta) = (\det \zeta)^{-\frac{1}{2}} \exp \left[ \frac{1}{2} \beta \eta^* \zeta^{-1} \beta + \beta (\mathbf{d}^* - \eta^* \zeta^{-1} \mathbf{d}) + \frac{1}{2} \mathbf{d} \eta^* \zeta^{-1} \mathbf{d} - \frac{1}{2} |\mathbf{d}|^2 \right]. \quad (10)$$

The photon distribution function of the *squeezed number state*  $|\mathbf{m}\rangle$  is described by the formula

$$\mathcal{P}_{\mathbf{n}} = |\det \zeta|^{-1} \exp \left[ \text{Re}(\mathbf{d} \eta^* \zeta^{-1} \mathbf{d}) - |\mathbf{d}|^2 \right] \frac{|H_{\mathbf{n}\mathbf{m}}^{\{\mathbf{R}\}}(\mathbf{L})|^2}{\mathbf{n}! \mathbf{m}!}. \quad (11)$$

Here  $\mathbf{m}$  is the label of the state, whereas  $\mathbf{n}$  is a discrete vector variable.  $2N \times 2N$ -matrix  $\mathbf{R}$  and  $2N$ -vector  $\mathbf{L}$  are expressed now in terms of blocks of matrix  $\Omega$  and vector  $\mathbf{d}$  as follows,

$$\mathbf{R} = \begin{pmatrix} \zeta^{-1} \eta & -\zeta^{-1} \\ -\zeta^{-1} & -\eta^* \zeta^{-1} \end{pmatrix}, \quad \mathbf{L} = \mathbf{R}^* \begin{pmatrix} -\zeta^{-1} \mathbf{d} \\ \mathbf{d}^* - \eta^* \zeta^{-1} \mathbf{d} \end{pmatrix}. \quad (12)$$

### 4 Even and Odd Coherent States

The one-mode even and odd coherent states have been introduced in Ref. [7]. The polymode even and odd coherent states have been introduced in Ref. [5]. The squeezed and correlated even and odd

coherent states have been introduced and studied in Ref.[4]. We will discuss the photon statistics of the light in these states which are also called Schrödinger cat states [8]. The multimode Schrödinger cat states are defined by the relation [5]

$$|A_{\pm}\rangle = N_{\pm}(|A\rangle \pm |-A\rangle), \quad (13)$$

where the multimode coherent state  $|A\rangle$  is

$$|A\rangle = |\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n\rangle = D(A)|0\rangle, \quad (14)$$

and  $D(A)$  is the multimode displacement operator creating coherent state from the vacuum. The normalization constants are

$$\begin{aligned} N_+ &= \frac{e^{\frac{|A|^2}{2}}}{2\sqrt{\cosh |A|^2}}, \\ N_- &= \frac{e^{\frac{|A|^2}{2}}}{2\sqrt{\sinh |A|^2}}, \end{aligned} \quad (15)$$

where complex number  $A$  has the form

$$|A|^2 = |\alpha_1|^2 + |\alpha_2|^2 + \dots + |\alpha_n|^2 = \sum_{m=1}^n |\alpha_m|^2. \quad (16)$$

The photon distribution function has the form [5]

$$\begin{aligned} P_+(n) &= \frac{|\alpha_1|^{2n_1} |\alpha_2|^{2n_2} \dots |\alpha_n|^{2n_n}}{(n_1!)(n_2!)\dots(n_n!)\cosh |A|^2}, \quad n_1 + n_2 + \dots + n_n = 2k, \\ P_-(n) &= \frac{|\alpha_1|^{2n_1} |\alpha_2|^{2n_2} \dots |\alpha_n|^{2n_n}}{(n_1!)(n_2!)\dots(n_n!)\sinh |A|^2}, \quad n_1 + n_2 + \dots + n_n = 2k + 1, \end{aligned} \quad (17)$$

and the photon means corresponding to these distributions are

$$\begin{aligned} \langle A_+ | n_i | A_+ \rangle &= |\alpha_i|^2 \tanh |A|^2, \\ \langle A_- | n_i | A_- \rangle &= |\alpha_i|^2 \coth |A|^2. \end{aligned} \quad (18)$$

The photon number dispersion matrix has the matrix elements

$$\begin{aligned} \sigma_{ik}^+ &= |\alpha_i|^2 |\alpha_k|^2 \operatorname{sech}^2 |A|^2 + |\alpha_i|^2 \tanh |A|^2 \delta_{ik}, \\ \sigma_{ik}^- &= -|\alpha_i|^2 |\alpha_k|^2 \operatorname{cosech}^2 |A|^2 + |\alpha_i|^2 \coth |A|^2 \delta_{ik}. \end{aligned} \quad (19)$$

## 5 Squeezed Schrödinger Cat States

Let us find out the photon statistics of squeezed polymode Schrödinger cat state labeled by the complex  $N$ -vector  $\beta$ . To do that let us define transition amplitude from the polymode squeezed and correlated state  $|\beta\rangle$  to the polymode photon number state  $|n\rangle$

$$T_n(\beta) = \mathcal{F}_0(\beta) \frac{1}{\sqrt{n!}} H_n^{\{\zeta^{-1}\eta\}}(\eta^{-1}[\beta - \mathbf{d}]), \quad (20)$$

where

$$\mathcal{F}_0(\beta) = (\det \zeta)^{-\frac{1}{2}} \exp \left[ \frac{1}{2} \beta \eta^* \zeta^{-1} \beta + \beta (\mathbf{d}^* - \eta^* \zeta^{-1} \mathbf{d}) + \frac{1}{2} \mathbf{d} \eta^* \zeta^{-1} \mathbf{d} - \frac{1}{2} |\mathbf{d}|^2 - \frac{1}{2} |\beta|^2 \right]. \quad (21)$$

Then the photon distribution function for polymode light in the squeezed Schrödinger cat state (even and odd) is given by the formula [4]

$$\mathcal{P}_n^\pm(\beta) = N_\pm^2(\beta) \left[ |T_n(\beta)|^2 + |T_n(-\beta)|^2 \pm (T_n^*(\beta) T_n(-\beta) + T_n(\beta) T_n^*(-\beta)) \right]. \quad (22)$$

If the shift parameter  $d = 0$  the formula is simplified

$$\mathcal{P}_{2k}^+(\beta) = 4N_+^2 \mathcal{P}_n(\beta), \quad (23)$$

if we have equality

$$\sum_{i=1}^N n_i = 2k, \quad (24)$$

and for even states

$$\mathcal{P}_{2k+1}^+(\beta) = 0, \quad (25)$$

if one has

$$\sum_{i=1}^N n_i = 2k + 1, \quad (26)$$

where  $\mathcal{P}_n(\beta)$  is given by the formula (3). For the light in the odd squeezed Schrödinger cat state the photon distribution is

$$\mathcal{P}_{2k+1}^-(\beta) = 4N_-^2 \mathcal{P}_n(\beta), \quad (27)$$

if the indices satisfy the equality (26) and

$$\mathcal{P}_{2k}^-(\beta) = 0, \quad (28)$$

if the indices satisfy relation (24). Thus the squeezed Schrödinger cat states if the shift parameter is equal to zero have highly oscillating distribution function. The influence of shift parameter decreases the oscillations of the distribution function. For Hermite polynomials the following sum rule may be found [3]

$$\begin{aligned} & \sum_{n_1=0}^{\infty} \cdots \sum_{n_N=0}^{\infty} \frac{\lambda_1^{n_1}}{n_1!} \frac{\lambda_2^{n_2}}{n_2!} \cdots \frac{\lambda_N^{n_N}}{n_N!} H_{n_1 n_2 \dots n_N}^{\{\mathbf{R}\}}(\mathbf{R}^{-1} \mathbf{z}) \\ &= [\det (\Lambda \Sigma_x \mathbf{R} + \mathbf{I}_{2N})]^{-\frac{1}{2}} \exp \left[ \frac{1}{2} \mathbf{z} (\Lambda \Sigma_x \mathbf{R} + \mathbf{I}_{2N})^{-1} \Sigma_x \Lambda \mathbf{z} \right]. \end{aligned} \quad (29)$$

Here  $\mathbf{z} = (z_1, z_2, \dots, z_{2N})$ , the  $2N \times 2N$  matrix  $\Sigma_x$  is the  $2N$ -dimensional analog of the Pauli matrix  $\sigma_x$ , and the diagonal  $2N \times 2N$  - matrix  $\Lambda$  has the matrix elements  $\lambda_j$  in  $j$ -th and  $(N+j)$ -th rows. Let us consider the one mode case. Then the formula for the photon distribution function in terms of Hermite polynomials of two variables may be expressed in terms of usual Hermite polynomials

$$\mathcal{P}_n = \mathcal{P}_0 \frac{(T^2 - 4d)^{\frac{n}{2}}}{(2T + 4d + 1)^n} \sum_{k=0}^n \left( \frac{4d - 1}{\sqrt{T^2 - 4d}} \right)^k \frac{n!}{[(n - k)!]^2 k!} \times \left| H_{n-k} \left( \frac{(T + 1)z + [\sigma_{pp} - \sigma_{qq} - 2i\sigma_{pq}]z^*}{\{(2T + 4d + 1)[\sigma_{pp} - \sigma_{qq} - 2i\sigma_{pq}]\}^{\frac{1}{2}}} \right) \right|^2. \quad (30)$$

Here the parameters  $\sigma_{pp}, \sigma_{qq}, \sigma_{pq}$  are matrix elements of the quadrature dispersion matrix  $M$ ,  $d$  is determinant of this matrix and  $T$  is the trace of the matrix. The complex number  $z$  is determined by the relation

$$z = \frac{1}{\sqrt{2}}(< q > + i < p >). \quad (31)$$

Formula (30) can be used also to illustrate the generalized uncertainty relation (for the Gaussian states). Indeed, it is obvious that the probability to find  $n$  photons must be nonnegative. On the other hand, all but one terms in the right-hand side of eq. (30) are positive independently on the concrete values of the parameters determining the quantum state. The only exception is the term

$$\left( \frac{4d - 1}{\sqrt{T^2 - 4d}} \right)^k.$$

Consequently, to guarantee the positiveness of the photon distribution function for all conceivable combinations of the parameters one should impose the restriction  $d \geq \frac{1}{4}$ . This inequality is the Schrödinger uncertainty relation (see, [6],[12]).

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